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Topological Solitons of the Nonlinear Schrödinger's Equation with Fourth Order Dispersion

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Abstract This paper obtains the topological 1-soliton solution of the nonlinear Schrödinger's equation, in a non-Kerr law media, with fourth order dispersion. An exact 1-soliton solution is obtained. The types of nonlinearity that are studied in this paper are Kerr law and power law.

Keywords Topological solitons · Integrability · Kerr law · Power law

1 Introduction

The nonlinear Schrödinger's equation (NLSE) plays a vital role in various areas of Physics, Applied Mathematics and Biochemistry [1–10]. It appears in the study of Nonlinear Fiber Optics, Plasma Physics, Fluid Dynamics and in α-helix proteins in Biochemistry [2] just to mention a few. One of the various classes of solutions of this NLSE is called solitons that is very important in the study of Physical and Biological Sciences. In this paper, a particular kind of solitons, called topological solitons, is going to be studied in the context of NLSE with fourth order dispersion (4OD). These topological solitons for the NLSE is particularly studied in the area of Nonlinear Optics in the context of dark optical solitons [6]. The NLSE with 4OD is going to be studied with Kerr and power law nonlinearity [9].

In Nonlinear Optics, the NLSE does not give correct prediction for pulse widths smaller than 1 picosecond. For example, in solid state solitary lasers, where pulses as short as 10 femtoseconds are generated, the approximation breaks down. Thus, quasi-monochromaticity is no longer valid and consequently higher order dispersion terms creep in. One needs to consider the higher order dispersion for performance enhancement along

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trans-oceanic and trans-continental distances. Also, for short pulse widths where the group velocity dispersion changes, within the spectral bandwidth of the signal, can no longer be neglected, one needs to take into account the presence of 4OD [9].

2 Mathematical Analysis

The dimensionless form of the NLSE with 4OD, in a non-Kerr law media is given by [1, 8, 9]

$$iq_t + aq_{xx} - bq_{xxxx} + cF(|q|^2)q = 0 \tag{1}$$

In (1), a, b and c are real numbers. If $b = 0$, (1) reduces to the regular NLSE with Kerr law nonlinearity. The coefficient of a represents the group velocity dispersion, while the coefficient of c represents the non-Kerr law nonlinearity. Also, the coefficient of b is the 4OD term. The solitons are the result of a delicate balance between dispersion and nonlinearity.

In (1), the function F is a real-valued algebraic function and it is necessary to have the smoothness of the complex function $F\left(|q|^2\right)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R^2, the function $F\left(|q|^2\right)q$ is k times continuously differentiable, so that [1]

$$F(|q|^2)q \in \bigcup_{m,n=1}^{\infty} C^k\left((-n,n)\times(-m,m); R^2\right) \tag{2}$$

Equation (1) is a nonlinear partial differential equation that is not integrable, in general. The non-integrability is not necessarily related to the nonlinear term in it. The soliton solution of (1), although not integrable, is assumed to be given in the amplitude-phase format as

$$q(x,t) = P(x,t)e^{i\phi(x,t)} \tag{3}$$

where

$$P(x,t) = Ag[B(x - vt)] \tag{4}$$

and

$$\phi(x,t) = -\kappa x + \omega t + \theta \tag{5}$$

For topological solitons the parameters A and B are free parameters while $\phi(x,t)$ is the phase of the soliton. Therefore κ is the soliton frequency, while ω is the wave number of the soliton and finally θ is the phase constant.

As mentioned earlier, (1) is not integrable by the classical method of IST. It is still possible to obtain a closed form 1-soliton solution of (1), once the law of nonlinearity is known. The method that will be used in this paper is the soliton ansatze. On substituting (3)–(5) into (1) yields

$$iq_t = \left(i\frac{\partial P}{\partial t} - P\frac{\partial \phi}{\partial t}\right)e^{i\phi} \tag{6}$$

$$q_{xx} = \left(\frac{\partial^2 P}{\partial x^2} - 2i\kappa\frac{\partial P}{\partial x} - \kappa^2 P\right)e^{i\phi} \tag{7}$$

$$q_{xxx} = \left(\frac{\partial^3 P}{\partial x^3} - 3i\kappa\frac{\partial^2 P}{\partial x^2} - 3\kappa^2\frac{\partial P}{\partial x} + i\kappa^3 P\right)e^{i\phi} \tag{8}$$

$$q_{xxxx} = \left(\frac{\partial^4 P}{\partial x^4} - 4i\kappa \frac{\partial^3 P}{\partial x^3} - 6\kappa^2 \frac{\partial^2 P}{\partial x^2} + 4i\kappa^3 \frac{\partial P}{\partial x} + \kappa^4 P \right) e^{i\phi} \tag{9}$$

Substituting (6)–(9) into (1) and equating the real and imaginary parts yields

$$\frac{\partial P}{\partial t} - 2\kappa(a + 2b\kappa^2)\frac{\partial P}{\partial x} + 4b\kappa \frac{\partial^3 P}{\partial x^3} = 0 \tag{10}$$

and

$$(\omega + a\kappa^2 + b\kappa^4)P - cP^3 - (a + 6b\kappa^2)\frac{\partial^2 P}{\partial x^2} + b\frac{\partial^4 P}{\partial x^4} = 0 \tag{11}$$

3 Kerr Law

For Kerr law nonlinearity

$$F(s) = s \tag{12}$$

Therefore (1) would reduce to

$$iq_t + aq_{xx} - bq_{xxxx} + c|q|^2 q = 0 \tag{13}$$

For topological solitons, with Kerr law nonlinearity, a proper choice for the function $g(x,t)$ would be [4]

$$g(x,t) = \tanh^p \tau \tag{14}$$

with

$$\tau = B(x - vt) \tag{15}$$

where the unknown exponent p will be determined during the course of derivation of the soliton solution to (1). Thus, from (14) and (15), (10) reduces to

$$\begin{aligned} &v(\tanh^{p-1}\tau - \tanh^{p+1}\tau) \\ &\quad + 2\kappa(a + 2b\kappa^2)(\tanh^{p-1}\tau - \tanh^{p+1}\tau) \\ &\quad - 4\kappa b B^2\big[(p-1)(p-2)\tanh^{p-3}\tau - (3p^2 - 3p + 2)\tanh^{p-1}\tau \\ &\quad + (3p^2 + 3p + 2)\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau\big] = 0 \end{aligned} \tag{16}$$

while (11) reduces to

$$\begin{aligned} &(\omega + a\kappa^2 + b\kappa^4)\tanh^p \tau \\ &\quad - pB^2(a + 6b\kappa^2)\big\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p \tau + (p+1)\tanh^{p+2}\tau\big\} \\ &\quad + bpB^4\big[(p-1)(p-2)(p-3)\tanh^{p-4}\tau \\ &\quad - \{2p^2 + (p-1)(p-2) + (p-2)(p-3)\}(p-1)\tanh^{p-2}\tau \\ &\quad + \{4p^3 + (p+1)^2(p+2) + (p-1)^2(p-2)\}\tanh^p \tau \\ &\quad - \{2p^2 + (p+1)(p+2) + (p+2)(p+3)\}(p+1)\tanh^{p+2}\tau \\ &\quad + (p+1)(p+2)(p+3)\tanh^{p+4}\tau\big] + cA^2\tanh^{3p}\tau = 0 \end{aligned} \tag{17}$$

while (15) would stay the same. Now, from (17) setting the exponents $3p$ and $p+4$ equal to one another gives

$$p = 2 \tag{18}$$

Also, noting that the functions $\tanh^{p+j}\tau$ for $j = -2, -4, 0, 2, 4$, are linearly independent, their respective coefficients in (17) must vanish. Therefore, this yields

$$A = \frac{\sqrt{15}(a + 6b\kappa^2)}{2b\sqrt{2}} \tag{19}$$

$$B = \sqrt{-\frac{a + 6b\kappa^2}{8b}} \tag{20}$$

$$\omega = -\left[136B^4 + \kappa^2(a + b\kappa^2) + 8B^2(a + 6b\kappa^2)\right] \tag{21}$$

Finally, applying the same strategy to (16), yields

$$v = -2\kappa(a + 2b\kappa^2) - 80b\kappa B^2 \tag{22}$$

From (19) and (20), one can conclude that the free parameters A and B of the soliton are related as

$$A = \sqrt{120bB^4} \tag{23}$$

From (20) and (23), it is possible to conclude that

$$a + 6b\kappa^2 < 0 \tag{24}$$

for the topological solitons to exist. Hence the topological 1-soliton solution of (13) is given by

$$q(x,t) = A\tanh^2[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \tag{25}$$

where the parameters A and B and the wave number along with the velocity of the soliton are given by (19)–(22) respectively.

4 Power Law

For power law nonlinearity,

$$F(s) = s^m \tag{26}$$

so that (1) would reduce to

$$iq_t + aq_{xx} - bq_{xxxx} + c\,|q|^{2m}q = 0 \tag{27}$$

where the parameter m, which is a real number, dictates the power law nonlinearity. In this case, starting with the same ansatze as in (14), (16) stays the same, while (17) changes to

$$\begin{aligned}
&(\omega + a\kappa^2 + b\kappa^4)\tanh^p\tau \\
&\quad - pB^2(a + 6b\kappa^2)\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau\right\} \\
&\quad + bpB^4\big[(p-1)(p-2)(p-3)\tanh^{p-4}\tau
\end{aligned}$$

$$
\begin{aligned}
&-\left\{2p^2+(p-1)(p-2)+(p-2)(p-3)\right\}(p-1)\tanh^{p-2}\tau\\
&+\left\{4p^3+(p+1)^2(p+2)+(p-1)^2(p-2)\right\}\tanh^{p}\tau\\
&-\left\{2p^2+(p+1)(p+2)+(p+2)(p+3)\right\}(p+1)\tanh^{p+2}\tau\\
&+(p+1)(p+2)(p+3)\tanh^{p+4}\tau\big]+cA^{2m}\tanh^{(2m+1)p}\tau=0
\end{aligned}
\tag{28}
$$

Now, from (28) setting the exponents $(2m+1)p$ and $p+4$ equal to one another gives

$$p=\frac{2}{m} \tag{29}$$

Again, from the linear independence of the functions $\tanh^{p+j}\tau$ for $j=-2, -4, 0, 2, 4$, gives

$$A=\left[\frac{(m+1)(m+2)(3m+2)(a+6b\kappa^2)^2}{16b^2(m^2-2m+2)^2}\right]^{\frac{1}{2m}} \tag{30}$$

$$B=\left[-\frac{m^2(a+6b\kappa^2)}{8b(m^2-2m+2)}\right]^{\frac{1}{2}} \tag{31}$$

$$\omega=-\frac{1}{m^4}\left[8bB^4(5m^2+12)+m^4\kappa^2(a+b\kappa^2)+8m^2B^2(a+6b\kappa^2)\right] \tag{32}$$

Finally, applying the same strategy to (16), yields

$$v=-2\kappa(a+2b\kappa^2)-\frac{8b\kappa B^2(m^2+3m+6)}{m^2} \tag{33}$$

From (30) and (31), one can conclude that the parameters A and B of the soliton are related as

$$A=\left[\frac{4b(m+1)(m+2)(3m+2)B^4}{m^4}\right]^{\frac{1}{2m}} \tag{34}$$

Thus from (31) and (34), it is possible to say that the topological solitons, for power law nonlinearity, will exist provided the same condition (24) holds. Hence, the 1-soliton solution of (27) is given by

$$q(x,t)=A\tanh^{\frac{2}{m}}[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)} \tag{35}$$

where the parameters A and B and the wave number along with the velocity of the soliton are given by (30)–(33) respectively.

5 Conclusions

In this paper, an exact topological 1-soliton solution to the NLSE with 4OD is obtained by the soliton ansatze. The governing equation is thus integrable although the Painleve test of integrability will fail. The types of nonlinearities that are considered in this paper are Kerr and power law. In future, other laws of nonlinearity will also be considered where a closed form soliton solution is available. Those investigations are under way.

In future, this NLSE will be studied along with its perturbation terms. This will also include the stochastic perturbation terms. The quasi-stationary soliton will be obtained in presence of such perturbation terms.

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